Saddle Point Contribution for an n-fold Complex-Valued Integral

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October 15, 2012

Abstract

Mathematical models of problems in the physical and dynamic sciences often lead to solutions represented by integrals. The method of steepest descent for integrals is a valuable tool for determining asymptotic approximations of such solutions. It is the method of choice when the integrand has an exponential part that is characterized by a complex-valued function multiplied by a "large" parameter. The method relies on Cauchy's theorem to deform the path of integration onto a contour (or contours) on which the exponential term causes the integrand to decay maximally from its value at "so-called" critical points. Application of Cauchy's theorem apparently limits the method to single integrals. Here, we show how the method can be applied to multiple integrals by iterating on the 1D method.

1 Introduction

In Bleistein and Gray, [2010], we required the leading term of the asymptotic expansion of a double integral with a complex exponent that depended on both variables of integration. To the best of the authors' knowledge, the derivation of the explicit asymptotic formula leading to the asymptotic expansion of this integral has not appeared in the open literature. Červený [2001] presents an outline of a method for the n-fold integral that relies on a theorem about symmetric complex-valued matrices. However, this author cannot find a proof of that result.

Having derived the leading term of the asymptotic expansion of the double integral in Bleistein and Gray, [2010], the asymptotic expansion of the n-fold integral becomes apparent. Here, we present the derivation of the leading term of the asymptotic expansion of that n-fold integral when the exponent has a simple saddle point in all n variables.

For a single integral, the method of choice for obtaining the asymptotic expansion is the method of steepest descent [Bleistein and Handelsman, 2010; Copson, 1965; Olver,1974; Wong, 2001]. That method relies on the ability to deform the path of integration in the complex plane onto a set of so-called paths of steepest descent. Such deformation relies on

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Cauchy's theorem for integration of analytic functions in the complex plane. Unfortunately, Cauchy's theorem does not extend to higher dimensions.

However, the method of iterated integration leads to the formula we seek The reader familiar with the method of multi-dimensional stationary phase will find the result here to be as expected.

We derive a result by using induction on n, the number of dimensions. That is, we assume that we have the "right" formula for the leading term in the asymptotic expansion in an $(n-1)$-fold integral and use that in the n-fold integral, by applying the 1D method of steepest descent to the integral in z_n. The only challenge in this approach is the determination of the second derivative in z_n at the saddle point, given the second derivative in z_{n-1}.

2 Background: $n = 1, 2$.

Here, we will present the first two cases of the n-fold integration, $n = 1, 2$. While the single integral case may be familiar to the reader, we use its review here to establish our notation and to introduce a modification of the canonical form of the integral that is presented in the literature.

2.1 $n = 1$.

We introduce the integral*

$$I(\omega) = \int f(z) \exp\{-\omega\Psi(z)\} dz. \tag{1}$$

We assume that the exponent has a saddle point that we can characterize by the equation

$$\frac{d\Psi}{dz} = 0, \quad z = z_{sad}. \tag{2}$$

Furthermore, we assume that the second derivative is nonzero at the saddle point,

$$\Psi_{zz} = \left.\frac{d^2\Psi}{dz^2}\right|_{z=z_{sad}} \neq 0; \tag{3}$$

We seek the leading order asymptotic expansion of this integral for "large" values of the parameter ω.

The Taylor expansion of Ψ near the saddle point has the form

$$\Psi(z) - \Psi(z_{sad}) = \frac{1}{2}\Psi_{zz}(z - z_{sad})^2 + \dots \ . \tag{4}$$

The direction of steepest descent in $z - z_{sad}$ at the saddle point is the direction in which this second order approximation is real and positive, which provides the direction of maximal exponential decay of the integrand.

*The minus sign in the exponent in (1) is what differs from the standard form of the literature. It will be seen below that this change simplifies the standard formula for the leading order asymptotic expansion.

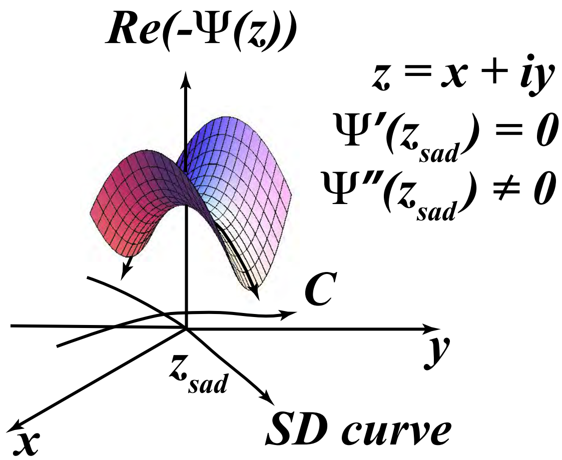

Figure 1: The surface $Re\{-\Psi(z)\}$ centered above the saddle point. The curve C is the original path of integration. The curve SD is the path of steepest descent away from the saddle point. It is the projection of the z plane of the path of maximal decrease on the surface $Re\{-\Psi(z)\}$ away from the saddle point on the surface.

Figure 2.1 shows the local behavior of $Re\{-\Psi(z)\}$ in the neighborhood of the saddle point. The local saddle-like behavior around the saddle point is depicted along with the curves through that saddle point along which the gradient of $Re\{-\Psi\}$ is maximal. The projections of those curves in the z-plane are then paths of steepest descent away from the saddle point.

We assume that the given curve of integration, C, can be deformed on to a pair of steepest descent curves with appropriate orientation, as shown in the figure. Along the SD curve of the figure, $\Psi(z) - \Psi(z_{sad})$ is real and increasing. That allows to apply LaPlace's method to obtain the leading order contribution from the asymptotic contribution from the saddle point.

In terms of the phases (denoted by arg in the complex variable literature) of the factors in the complex product appearing on the right side of the last equation, this condition becomes,

$$\arg(\Psi_{zz}) + 2\arg(z - z_{sad}) = 0,\ 2\pi,\ \ldots\ . \tag{5}$$

We expect that the direction of choice will be a rotation of the contour of integration through an acute angle. Thus from the two unique choices of direction here we choose

$$\arg(z - z_{sad}) = -\arg(\Psi_{zz})/2, \tag{6}$$

such that the oriented direction with this angle makes an acute angle with the original direction of the path of integration.

The method of steepest descent relies on deforming the path of integration onto a path where the exponential difference $\Psi(z) - \Psi(z_{sad})$ remains real and increasing and then applying the more basic Laplace method [Bleistein, 1984] to the integral on that steepest descent contour.

Application of the formula for evaluation of the integral of equation (1) by the method of steepest descent[†] [Bleistein, 1984, equation (7.3.11)] with positive ω leads to the following.

$$\begin{aligned} I(\omega) &\sim \sqrt{\frac{2\pi}{\omega|\Psi_{zz}|}} f(z_{sad}) \exp\left\{-\omega\Psi(z_{sad}) - i\arg(\Psi_{zz})/2\right\} \\ &= \sqrt{\frac{2\pi}{\omega\Psi_{zz}}} f(z_{sad}) \exp\left\{-\omega\Psi(z_{sad})\right\}. \end{aligned} \tag{7}$$

Notice that by defining the integrand on the right hand side of equation (1) for $I(\omega)$ with a minus sign in the exponent, the phase adjustment in the first line here provides exactly the right factor so that the denominator can be expressed as a specific branch of the square root of the second derivative. If we had defined the exponent without that minus sign, then we would have needed an additional phase shift of $\pm\pi/2$ in the right hand exponents of this last equation.

2.2 Iterated steepest descent in two dimensions

This is an extension not found in texts on the method of steepest descent. For this purpose, we introduce indexed variables and consider the integral

$$I(\omega) = \int f(\boldsymbol{z}) \exp\{-\omega\Psi(\boldsymbol{z})\} dz_1 dz_2, \quad \boldsymbol{z} = (z_1, z_2). \tag{8}$$

Here, we will allow for deformations into the complex z_1-plane to obtain the asymptotic expansion of the integral with respect to z_1 and similarly for z_2. Both functions in the integrand are complex-valued.

We assume that the exponent has a simple saddle point in both variables; it is characterized by the equation

$$\nabla_z \Psi(\boldsymbol{z}) = \boldsymbol{0}, \quad \boldsymbol{z} = \boldsymbol{z}_{sad} = (z_{1sad}, z_{2sad}). \tag{9}$$

Furthermore, we assume that the Hessian matrix, the matrix of second derivatives, of Ψ is non-singular at this saddle point; that is

$$\det[\boldsymbol{\Psi}_2] \neq 0, \quad \boldsymbol{z} = \boldsymbol{z_{sad}}; \boldsymbol{\Psi}_2 = \left[\frac{\partial^2 \Psi(\boldsymbol{z}_{sad}}{\partial z_i \partial z_j}\right], i, j = 1, 2. \tag{10}$$

We will also assume that

$$\left.\frac{\partial^2 \Psi}{\partial z_1^2}\right|_{\boldsymbol{z}=\boldsymbol{z_{sad}}} \neq 0. \tag{11}$$

If this were not the case, we could rotate the coordinate system to make this so; such rotations use matrices with determinant equal to one, so that they do not affect the final formula for the asymptotic expansion of $I(\omega)$ in equation (8).

[†]There we use $\lambda = -\omega$ as the large parameter.

Let us consider the integration in z_1 alone in equation (8). From equation (10), that integral has a saddle point when

$$\frac{\partial \Psi(z_1, z_2)}{\partial z_1} = 0. \tag{12}$$

By the assumption of the existence of the "simple" saddle point in equation (10), we know that this equation has a solution for $z_2 = z_{2sad}$, at which point, $z_1 = z_{1sad}$. By the implicit function theorem, equation (12) has a unique solution in the neighborhood of $z_1 = z_{1sad}$ by virtue of the assumption of equation (11) that the second derivative with respect to z_1 is nonzero at the saddle point. Therefore we can write

$$z_1 = Z_1(z_2), \quad \frac{\partial \Psi(Z_1(z_2), z_2)}{\partial z_1} \equiv 0 \tag{13}$$

in some neighborhood of $\boldsymbol{z} = \boldsymbol{z}_{sad}$.

We can now write down the leading order asymptotic contribution with respect to z_1 of the iterated integral $I(\omega)$ in equation (8) by using the formula of equation (7) for that asymptotic expansion in one variable:

$$I(\omega) = \sqrt{\frac{2\pi}{\omega}} \int \frac{f(Z_1(z_2), z_2)}{\sqrt{\Psi_{z_1 z_1}(Z_1(z_2), z_2)}} \exp\{-\omega \Psi(Z_1(z_2), z_2)\} dz_2. \tag{14}$$

Now, let us consider the integration in z_2. We write down the first derivative with respect to z_2 of this redefined exponent Ψ by applying the chain rule to deal with the dependence of the first variable z_1 on z_2.

$$\frac{d\Psi(Z_1(z_2), z_2)}{dz_2} = \frac{\partial \Psi}{\partial z_1}\frac{dZ_1}{dz_2} + \frac{\partial \Psi}{\partial z_2}. \tag{15}$$

The first term here is identically equal to zero as a result of the stationarity condition in equation (13). Therefore, let us rewrite the first derivative here accordingly and then write down the second derivative, as well.

$$\begin{aligned} \frac{d\Psi(Z_1(z_2), z_2)}{dz_2} &= \frac{\partial \Psi}{\partial z_2}, \\ \frac{d^2\Psi(Z_1(z_2), z_2)}{dz_2^2} &= \frac{\partial^2 \Psi}{\partial z_2 \partial z_1}\frac{dZ_1}{dz_2} + \frac{\partial^2 \Psi}{\partial z_2^2} \end{aligned} \tag{16}$$

From the first line here, we see that the condition that the total derivative with respect to z_2 of this new exponent be equal to zero is exactly the condition that the partial derivative with respect to z_2 be equal to zero. This is just the requirement that the second component of the gradient of Ψ be equal to zero in equation (9). We then conclude that the saddle point occurs at $z_2 = z_{2sad}$ for which value we also have $z_1 = z_{1sad}$. In summary, the dual saddle point of $\Psi(z_1, z_2)$ obtained by setting its gradient equal to zero is the same as the simultaneous saddle point in two separate variables determined by iterated application of the method of steepest descent.

Now we must evaluate the second derivative of Ψ in equation (16) at the saddle point. To this end, we must first express the derivative of $Z_1(z_2)$ in terms of derivatives of Ψ. The function $Z_1(z_2)$ is defined implicitly in equation (13). We differentiate that equation with respect to z_2:

$$\frac{\partial^2\Psi(Z_1(z_2), z_2)}{\partial z_1^2}\frac{dZ_1(z_2)}{dz_2} + \frac{\partial^2\Psi(Z_1(z_2), z_2)}{\partial z_1 \partial z_2} \equiv 0. \tag{17}$$

The coefficient of the derivative of Z_1 is nonzero at the saddle point—equation (11)—so we can divide by it and conclude that

$$\frac{dZ_1(z_2)}{dz_2} = -\frac{\partial^2\Psi(Z_1(z_2), z_2)}{\partial z_1 \partial dz_2}\left[\frac{\partial^2\Psi(Z_1(z_2), z_2)}{\partial z_1^2}\right]^{-1}. \tag{18}$$

We substitute this value of the first derivative into the second line of equation (15) to obtain the representation we seek for the second derivative of Ψ with respect to z_2 evaluated at $\boldsymbol{z}_{2sad}$.

$$\begin{aligned}\frac{d^2\Psi(Z_1(z_2), z_2)}{dz_2^2} &= \left[\frac{\partial^2\Psi}{\partial z_1^2}\frac{\partial^2\Psi}{\partial z_2^2} - \left(\frac{\partial^2\Psi}{\partial z_1 \partial z_2}\right)^2\right]\left[\frac{\partial^2\Psi(Z_1(z_2), z_2)}{\partial z_1^2}\right]^{-1} \\ &= \det[\boldsymbol{\Psi}_2]\left[\frac{\partial\Psi(Z_1(z_2), z_2)}{\partial z_1^2}\right]^{-1}\Bigg|_{\boldsymbol{z}=\boldsymbol{z}_{sad}}.\end{aligned} \tag{19}$$

We again apply the asymptotic expansion formula of equation (7) to the integral of equation (14) and find that

$$I(\omega) \sim \frac{2\pi}{\omega}\frac{f(\boldsymbol{z}_{sad})}{\sqrt{\det[\boldsymbol{\Psi}_2]}}\exp\{-\omega\Psi(\boldsymbol{z}_{sad})\}. \tag{20}$$

The structure of this result echoes the formula for the 2D method of stationary phase [Bleistein, [1984], equation (2.8.23). It also suggests the $n-$dimensional generalization as well as the conditions on the $n \times n$ Hessian.

3 The n-dimensional case

We now consider the n-fold generalization of the the 2D integral of equation (8):

$$I_n(\omega) = \int f(\boldsymbol{z})\exp\{-\omega\Psi(\boldsymbol{z})\}dz_1 dz_2 \ldots dz_n, \quad \boldsymbol{z} = (z_1, z_2, \ldots z_n). \tag{21}$$

We assume that the exponent has a simple saddle point in all variables that we characterize by the equation

$$\nabla_z\Psi(\boldsymbol{z}) = \boldsymbol{0}, \quad \boldsymbol{z} = \boldsymbol{z}_{sad} = (z_{1sad}, z_{2sad}, \ldots z_{nsad}). \tag{22}$$

Furthermore, to make the saddle point simple we assume that the Hessian matrix, the matrix of second derivatives, of Ψ_n is non-singular at this saddle point; that is

$$\det[\boldsymbol{\Psi}_n] \neq 0, \quad \boldsymbol{\Psi}_n = \left[\frac{\partial^2\Psi}{\partial z_i \partial z_j}\right]\Bigg|_{\boldsymbol{z}=\boldsymbol{z}_{sad}}, i, j = 1, 2, \ldots n. \tag{23}$$

We will use iteration and induction to derive a steepest descent formula for the n-dimensional integral of equation (21). Each step of that iteration requires a nonsingular Hessian matrix. So, for each $k, 1 \leq k \leq n$ we require,

$$\det[\boldsymbol{\Psi}_k] \neq 0, \quad \boldsymbol{\Psi}_k = \left[\frac{\partial^2 \Psi}{\partial z_i \partial z_j} \right] \bigg|_{\boldsymbol{z}=\boldsymbol{z}_{sad}}, i, j = 1, 2, \ldots, k. \tag{24}$$

We use induction to derive the asymptotic expansion of the integral of order n from the integral of order $n-1$. That is, we assume that

$$I_{n-1}(\omega) \sim \left[\frac{2\pi}{\omega} \right]^{(n-1)/2} \frac{f(\boldsymbol{Z}_{sad}(z_n), z_{z_n})}{\sqrt{\det[\boldsymbol{\Psi_{n-1}}]}} \exp\{-\omega \Psi(\boldsymbol{Z}_{sad}(z_n), z_n)\}. \tag{25}$$

We wish to verify that

$$I_n(\omega) \sim \left[\frac{2\pi}{\omega} \right]^{n/2} \frac{f(\boldsymbol{z}_{sad})}{\sqrt{\det[\boldsymbol{\Psi_n}]}} \exp\{-\omega \Psi(\boldsymbol{z}_{sad})\}. \tag{26}$$

Note that the derived asymptotic expansions (7) and (20) provide the starting point for the induction at $n = 2$. In the n-fold integral of equation (21), we apply the assumed asymptotic expansion of the integral of order $n-1$ as stated in equation (25) to obtain the following:

$$I_n(\omega) \sim \left[\frac{2\pi}{\omega} \right]^{(n-1)/2} \int \frac{f(\boldsymbol{Z}_{sad}(z_n), z_{z_n})}{\sqrt{\det[\boldsymbol{\Psi_{n-1}}(\boldsymbol{Z}_{sad}(z_n), z_n)]}} \exp\{-\omega \Psi(\boldsymbol{Z}_{sad}(z_n), z_n)\} dz_n. \tag{27}$$

Here,

$$\boldsymbol{Z}_{sad}(z_n) = (Z_{1sad}(z_n), Z_{2sad}(z_n) \ldots, Z_{\{n-1\}sad}(z_n)). \tag{28}$$

Furthermore, in the neighborhood of $\boldsymbol{z} = \boldsymbol{z}_{sad}$,

$$\nabla_z \Psi(\boldsymbol{Z}_{sad}(z_n), z_n) \equiv \left(\frac{\partial \Psi(\boldsymbol{Z}_{sad}(z_n), z_n)}{\partial z_1}, \frac{\partial \Psi(\boldsymbol{Z}_{sad}(z_n), z_n)}{\partial z_2}, \ldots, \frac{\partial \Psi(\boldsymbol{Z}_{sad}(z_n), z_n)}{\partial z_{n-1}} \right) \equiv \boldsymbol{0}, \tag{29}$$

with ∇_z here being $n-1$ dimensional. We know from the existence of an n-dimensional saddle point, equation (22), that the $n-1$ equations in $n-1$ unknowns, (29), *does* have a solution when $z_n = z_{nsad}$, that solution being $\boldsymbol{z} = \boldsymbol{z}_{sad}$.

We propose now to apply the method of steepest descent to the integral for $I_n(\omega)$ in equation (27). To begin, we write down the first derivative of the exponent $\Psi(\boldsymbol{Z}_{sad}(z_n), z_n)$ in that equation:

$$\frac{d\Psi}{dz_n} = \sum_{k=1}^{n-1} \frac{\partial \Psi}{\partial z_k} \frac{dZ_k}{dz_n} + \frac{\partial \Psi}{\partial z_n} = \frac{\partial \Psi}{\partial z_n}. \tag{30}$$

The entire sum in this equation is zero because each z_k derivative of Ψ, $k = 1, \ldots, n-1$, is zero (equation (29)). Therefore the total derivative with respect to z_n is just the partial derivative with respect to z_n, the last independent variable in Ψ.

The saddle point in z_n is determined by setting the derivative $d\Psi/dz_n$ in equation (30) equal to zero. Since this total derivative reduces to the partial derivative, $\partial \Psi/\partial z_n$, the entire

n dimensional gradient of Ψ must be zero. That zero occurs at the point $\boldsymbol{z} = \boldsymbol{z}_{sad}$, equation (22). Hence, the saddle point in all n variables determined by iteration is exactly the saddle point $\boldsymbol{z}_{nsad}$ of $\Psi(\boldsymbol{z})$ in equation (22).

The asymptotic expansion formula in one dimension, given in equation (7), requires that we determine the second derivative, $d^2\Psi/dz_n^2$ at the saddle point. Starting from equation (30) for the first derivative of Ψ, we calculate[‡]

$$\frac{d^2\Psi}{dz_n^2} = \frac{d}{dz_n}\left[\frac{\partial\Psi}{\partial z_n}\right] = \sum_{k=1}^{n-1} \frac{\partial^2\Psi}{\partial z_n \partial z_k}\frac{dZ_k}{dz_n} + \frac{\partial^2\Psi}{\partial z_n^2}. \tag{31}$$

The derivatives of the functions Z_k in this equation can be expressed in terms of derivatives of Ψ. To do so, let us first consider the identity for the $n-1$ components of the gradient in equation (29) and differentiate each component of the gradient with respect to z_n:

$$\frac{d}{dz_n}\left[\frac{\partial\Psi}{\partial z_m}\right] = \sum_{j=1}^{n-1} \frac{\partial^2\Psi}{\partial z_m \partial z_j}\frac{dZ_j}{dz_n} + \frac{\partial^2\Psi}{\partial z_m \partial z_n} \equiv 0, \quad m = 1, 2, \ldots, n-1. \tag{32}$$

The coefficients of the set of unknowns—$dZ_j/dz_n, j = 1, \ldots n-1$—here are just the elements of the $(n-1)\times(n-1)$ Hessian matrix,

$$\boldsymbol{\Psi}_{(n-1)} = \left[\frac{\partial^2\Psi}{\partial z_m \partial z_j}\right], \quad m, j = 1, \ldots, n-1. \tag{33}$$

To solve this system of equations we first multiply through by the inverse of this Hessian matrix,

$$\boldsymbol{\Psi}_{(n-1)}^{-1} = \frac{1}{\det\left(\boldsymbol{\Psi}_{(n-1)}\right)}\left[\mathrm{cof}\left(\frac{\partial^2\Psi}{\partial z_k \partial z_m}\right)\right], \quad m, k = 1, \ldots, n-1. \tag{34}$$

In this equation cof denotes the cofactor of the element in the mth row and kth column; this is the determinant of an $(n-2)\times(n-2)$ matrix, obtained from $\boldsymbol{\Psi}_{(n-1)}$ by eliminating the mth row and kth column. (Because of the symmetry of the matrix $\boldsymbol{\Psi}_{(n-1)}$, there is no need to introduce the *transpose* of the matrix before computing the cofactor.)

We multiply equation (32) by the inverse in equation (34) to find that

$$\frac{dZ_k}{dz_n} + \frac{1}{\det\left(\boldsymbol{\Psi}_{(n-1)}\right)}\sum_{m=1}^{n-1}\mathrm{cof}\left(\frac{\partial^2\Psi}{\partial z_k \partial z_m}\right)\frac{\partial^2\Psi}{\partial z_m \partial z_n} \equiv 0, \quad k = 1, \ldots, n-1. \tag{35}$$

Solving here for dZ_k/dz_n,

$$\frac{dZ_k}{dz_n} = -\frac{1}{\det\left(\boldsymbol{\Psi}_{(n-1)}\right)}\sum_{m=1}^{n-1}\mathrm{cof}\left(\frac{\partial^2\Psi}{\partial z_k \partial z_m}\right)\frac{\partial^2\Psi}{\partial z_m \partial z_n} \equiv 0, \quad k = 1, \ldots, n-1. \tag{36}$$

[‡]Compare with equation (16).

Recall that the purpose of this last calculation was to express $d^2\Psi/dz_n^2$ in equation (31) totally in terms of derivatives of Ψ. We substitute the expressions for $dZ_k dz_n$ in equation (36) into equation (31) to achieve this end. We then find that

$$\frac{d^2\Psi}{dz_n^2} = \frac{1}{\det\left(\mathbf{\Psi}_{(n-1)}\right)}\left[\frac{\partial^2\Psi}{\partial z_n^2}\det\left(\mathbf{\Psi}_{(n-1)}\right) - \sum_{k=1}^{n-1}\sum_{m=1}^{n-1}\frac{\partial^2\Psi}{\partial z_n \partial z_k}\mathrm{cof}\left(\frac{\partial^2\Psi}{\partial z_k \partial z_m}\right)\frac{\partial^2\Psi}{\partial z_m \partial z_n}\right] \tag{37}$$

In the Appendix, we show that the expression in square brackets is actually $\det(\mathbf{\Psi}_n)$, which is the determinant of the $n \times n$ Hessian of equation (23), so that

$$\frac{d^2\Psi}{dz_n^2} = \frac{\det(\mathbf{\Psi}_n)}{\det\left(\mathbf{\Psi}_{(n-1)}\right)}. \tag{38}$$

We can now use the formula of equation (7) for the leading order contribution from a saddle point in a single integral. The second derivative in that formula is given by equation (38). We apply these results to the integral for $I_n(\omega)$ in equation (27). The result is the asymptotic expansion of $I_n(\omega)$ in equation (26), which is what we wanted to verify.

This derivation is an alternative to one presented by Červený [1982] and Červený [2001]. It has the additional feature of providing a mechanism for determining the proper branch of the complex-valued square root in the denominator of the final formula.

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Appendix: Verification of Equation (38)

The claim of equation (38) that the second derivative of Ψ with respect to z_n will be verified here.

Examination of equation (37) for that second derivative reveals that what we really need to verify is that

$$\det\left(\mathbf{\Psi}_n\right) = \frac{\partial^2\Psi}{\partial z_n^2}\det\left(\mathbf{\Psi}_{(n-1)}\right) - \sum_{k=1}^{n-1}\sum_{m=1}^{n-1}\frac{\partial^2\Psi}{\partial z_n \partial z_k}\mathrm{cof}\left(\frac{\partial^2\Psi}{\partial z_k \partial z_m}\right)\frac{\partial^2\Psi}{\partial z_m \partial z_n}. \tag{A-1}$$

It is helpful for the moment to think of the double sum here as an $(n-1)\times(n-1)$ matrix multiplied by coefficients arising from the the missing nth row and n column. The elements of the cofactor matrix are themselves determinants of $(n-2)\times(n-2)$ matrices that exclude the mth row and kth column of the $(n-1)\times(n-1)$ matrix of cofactors.

Let us fix k for the moment and consider only the sum on m. This is a sum over m rows of the nth column of the $n \times n$ Hessian. The sum on m then becomes the determinant of a matrix on size $(n-1)\times(n-1)$. If we think of interchanging the mth column in the cofactor matrix with the nth column, we can verify that the sum over m is just the determinant,

$$\sum_{m=1}^{n-1}\mathrm{cof}\left(\frac{\partial^2\Psi}{\partial z_k \partial z_m}\right)\frac{\partial^2\Psi}{\partial z_m \partial z_n} = -\det\left(\frac{\partial^2\Psi}{\partial z_k \partial z_n}\right). \tag{A-2}$$

The minus sign here arises from the interchange of two columns.

This revises the right side in equation (A-2) for $\det\left(\mathbf{\Psi}_n\right)$ to

$$\det\left(\mathbf{\Psi}_n\right) = \frac{\partial^2\Psi}{\partial z_n^2}\det\left(\mathbf{\Psi}_{(n-1)}\right) + \sum_{k=1}^{n-1}\frac{\partial^2\Psi}{\partial z_n \partial z_k}\det\left(\frac{\partial^2\Psi}{\partial z_k \partial z_n}\right). \tag{A-3}$$

If the sum in this equation ranged from 1 to n, then this would be the expansion by the *nth* row of the $n \times n$ Hessian. In fact, the only term missing from that sum is the first term on the left.

Consequently, we have confirmed out conjecture in equation (A-1) that the the two sides of that equation agree. In turn, this confirms the claim in equation (38) that the second derivative of exponent is indeed determinant of the Hessian of order n divided by the determinant of the Hessian of order $n-1$.

The asymptotic formula for $I_n(\omega)$ in equation (26) is now verified.